

The Categorified Heisenberg Algebra I: A Combinatorial Representation

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Abstract

We give an introductory account of Khovanov’s categorification of the Heisenberg algebra, and construct a combinatorial model for it in a 2-category of spans of groupoids. We also treat a categorification of $U(\mathfrak{sl}_n)$ in a similar way. These give rise to standard representations on vector spaces through a linearization process.

1 Introduction

1.1 Overview

Our objective in this paper is to describe a combinatorial model of Khovanov’s categorification of the Heisenberg algebra [13], using a natural construction based on the groupoidification programme of Baez and Dolan [1, 2]. This gives rise to a narrative which serves to ‘explain’ the structure of the categorified algebra in terms of the combinatorics of finite sets. It is also fundamental, giving rise to known linear representations via a canonical process of linearization. This account is one part of a theory of representations of the categorified Heisenberg algebra in terms of free symmetric algebraic structures, described in a forthcoming companion article [18].

The model of the categorified Heisenberg algebra which we describe is based on a groupoidification of the quantum harmonic oscillator, a simple quantum-mechanical system. This system has an infinite-dimensional Hilbert space of states called *Fock space*, which carries an action of a *Heisenberg algebra*. The simplest of these, describing a quantum harmonic oscillator with a single degree of freedom, is the free complex algebra on generators \mathbf{a}^\dagger and \mathbf{a} , called the *creation* and *annihilation* operators respectively, modulo the commutation relation

$$\mathbf{a}\mathbf{a}^\dagger - \mathbf{a}^\dagger\mathbf{a} = 1. \quad (1)$$

The action on Fock space involves unbounded operators, and equation (1) is only required to hold on a dense domain.

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The Fock space for a single oscillator has a standard basis of states, called the Fock basis, which are labelled by nonnegative integer *energy levels*. Elements of this basis represent states in which the system contains an integer number of energy quanta, interpreted in a quantum field theory setting as ‘particles’. The creation and annihilation operators act on this basis to increase or decrease the number of particles by one.

This description is already understood to be a ‘shadow’, or *decategorification*, of a richer perspective in which the actual sets of particles are themselves the relevant mathematical objects [1, 2, 19]. Thus, for instance, rather than merely describing a change in particle number, one can instead consider the maps between sets of particles which *witness* that change. This makes room for more structure, so that one can consider the action of the symmetric group which corresponds to physically permuting the particles, an act which has no nontrivial mathematical representation in the case of ordinary Fock space. The usual Fock space picture is recovered in the decategorification (or “degroupoidification”) of this structure.

The categorified Heisenberg algebra captures the interesting mathematical statements that can be made in this richer setting. The new contribution here is the interpretation of these statements in terms of the combinatorics of finite sets, giving new insight into the categorified Heisenberg algebra. We demonstrate this without formal theorems, emphasizing with an emphasis on developing a good intuition for the structures at hand.

These structures have an abstract mathematical description which makes them more general than any one model. In particular, there are applications in computer science [5, 6, 7] for which a Fock space-like structure represents an unlimited number of copies of some logical resource; the constructions of this paper are relevant to categorifications of this scenario. We will explore this more general perspective in a more technical article [18], where we show how representations of the categorified Heisenberg algebra arise generically on free symmetric pseudomonoid structures internal to monoidal 2-categories with sufficiently good properties.

1.2 Khovanov’s categorification

More technically, to *categorify* a complex algebra means to find a monoidal category \mathbf{C} with coproducts, such that the algebra can be recovered as the complexification of the monoid of isomorphism classes of objects of \mathbf{C} , with the vector space structure on the algebra arising from the coproduct structure in \mathbf{C} . This can be seen as the composition of two mathematical processes:

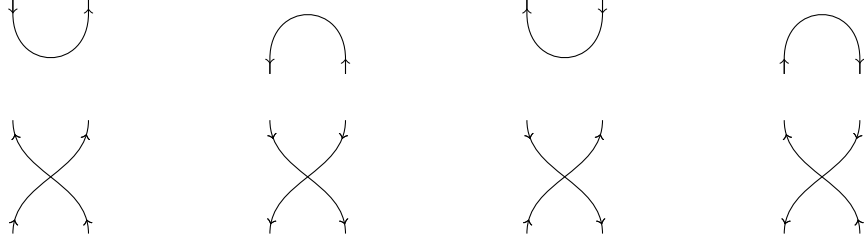
$$[\text{Monoidal Categories}] \xrightarrow{K_0} [\text{Rings}] \xrightarrow{\mathbb{C} \otimes -} [\text{Algebras}] \quad (2)$$

Khovanov has given a categorification of the Heisenberg algebra in this sense [13], part of a broader programme which includes the categorification of quantum groups [14, 26]. In general, this programme involves the construction of monoidal categories whose morphisms are classes of diagrams, sometimes with decorations of various kinds, modulo certain topological identifications.

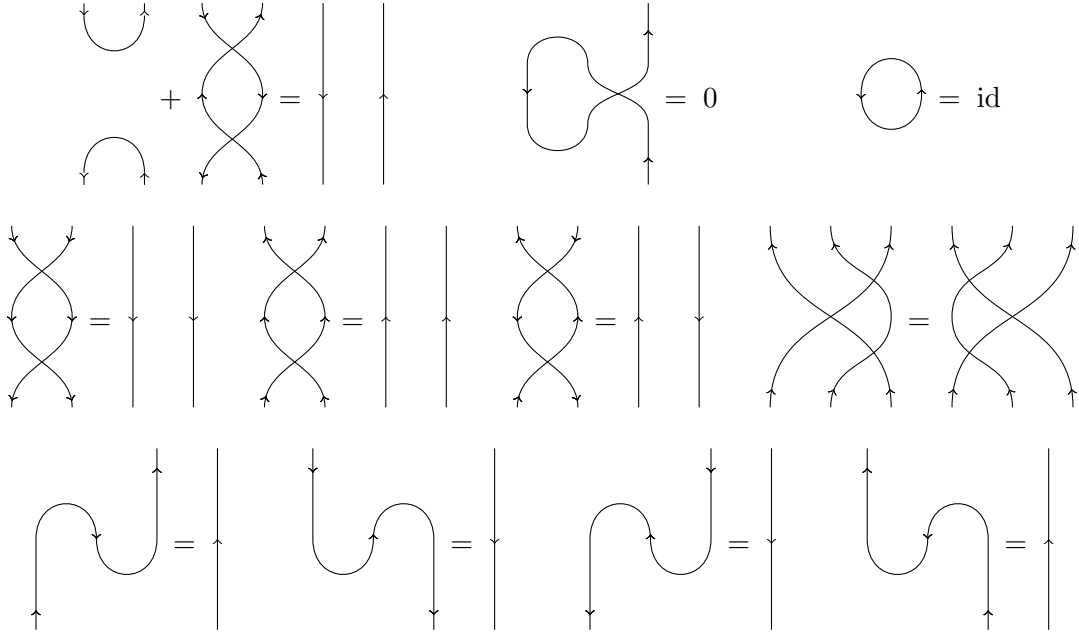
Khovanov’s categorification of the Heisenberg algebra takes the form of a monoidal category \mathbf{H}' with a zero object and biproducts, with generating objects Q_+ and Q_- depicted as upwards- and downwards-pointing strands respectively:

$$\begin{array}{c} \downarrow \\ Q_+ \end{array} \qquad \begin{array}{c} \downarrow \\ Q_- \end{array}$$

Tensor product is represented by horizontal juxtaposition. The generating morphisms have the following graphical representations:



The following equations are then imposed between composites of these generating morphisms:



These equations imply an isomorphism

$$Q_- \otimes Q_+ \simeq Q_+ \otimes Q_- \oplus I \quad (3)$$

where I is the monoidal unit object, giving a categorification of the Heisenberg algebra relation (1).

Khovanov goes on to show that this has a representation on a monoidal category whose objects are bimodules describing restriction and induction of representations of symmetric groups. As we will see, this amounts to a representation on *2-vector spaces* in the sense of Kapranov and Voevodsky, giving a 2-functor:

$$\mathbf{H}' \xrightarrow{K} \mathbf{2Vect} \quad (4)$$

where we see \mathbf{H}' as a 2-category with one object. This takes the single object of \mathbf{H}' to a 2-vector space $\coprod_n \text{Rep}(S_n)$, the coproduct of the categories of representations of the symmetric groups, which plays the role of a categorified Fock space.

We will restrict our attention to this monoidal category \mathbf{H}' of diagrams in Section 2, when we give them a combinatorial interpretation. However, in fact, the categorification of the full Heisenberg algebra with multiple generators a_n requires somewhat more. The

generators a_n are represented in the categorification by “symmetric products” of the basic objects Q_+ and Q_- . In the diagrammatic categorification, this is done formally by completing \mathbf{H}' to a category called \mathbf{H} , in which all the required subobjects of Q_+^n exist. In Section 3, we will turn to this in more detail, and in particular 3.4 discusses how the particular symmetrizer subobjects which appear in this completion occur naturally in $\mathbf{2Vect}$ as sub-functors, and describes some of them in detail.

1.3 Groupoidification

The concrete model of \mathbf{H}' we find arises from a seemingly different approach to categorifying the Heisenberg algebra, based on the *groupoidification* program of Baez and Dolan [2]. A *groupoid* is a category with all morphisms invertible. The central objects of study in the groupoidification programme are *spans of groupoids*, diagrams of the form

$$\begin{array}{ccc} & \mathbf{X} & \\ G \swarrow & & \searrow F \\ \mathbf{B} & & \mathbf{A} \end{array} \quad (5)$$

where \mathbf{A} , \mathbf{B} and \mathbf{X} are groupoids, and F and G are functors. There has been extensive work within category theory on constructions involving spans [4]. The most immediately important fact is that spans of groupoids can be organized into a 2-category $\mathbf{Span}(\mathbf{Gpd})$. The main construction of this paper is a 2-functor

$$\mathbf{H}' \xrightarrow{C} \mathbf{Span}(\mathbf{Gpd}), \quad (6)$$

where again we view \mathbf{H}' as a 2-category with one object. This yields a representation of Khovanov’s categorification of the Heisenberg algebra in terms of spans of groupoids.

An important interpretation of groupoidification comes from physics: groupoids represent *physical symmetries*, and spans represent *spaces of histories*. The idea is that configuration spaces of physical systems can be represented as groupoids in a way that usefully encodes their symmetries, and that spans of groupoids encode ways in which these states and their symmetries can be transformed via physical processes. In our example of the harmonic oscillator, these configurations are the integer-valued energy eigenstates. Then a span represents a space of *histories*, with its source and target maps picking out the starting and ending configurations. Furthermore, these are not just set-maps of the objects (histories and configurations), but functors, which also describe how symmetries of histories act on starting and ending configurations.

The single object of \mathbf{H}' is mapped by C to the groupoid \mathbf{S} of finite sets and bijections. Spans involving this groupoid have an elegant interpretation in terms of the combinatorics of finite sets, thanks to work of Joyal [12] and Baez and Dolan [1]. The resulting combinatorial interpretation of our representation gives us a new perspective on the categorified Heisenberg algebra.

A construction called *2-linearization* [20] converts a span of groupoids to a 2-linear map between 2-vector spaces. This resulting 2-linear map can be thought of as being ‘accounted for’ by the underlying span of groupoids, giving rise to a useful combinatorial perspective on the mathematics. The 2-linearization process gives a 2-functor of the following type:

$$\mathbf{Span}(\mathbf{Gpd}) \xrightarrow{\Lambda} \mathbf{2Vect} \quad (7)$$

Composing this functor with our representation (6) gives us a linear representation of the categorified Heisenberg algebra:

$$\mathbf{H}' \xrightarrow{C} \mathbf{Span}(\mathbf{Gpd}) \xrightarrow{\Lambda} \mathbf{2Vect} \quad (8)$$

We will see that this reproduces Khovanov's representation (4), and hence can be seen as giving a concrete combinatorial 'explanation' of his construction.

In Section 4 we extend these ideas to produce a groupoidification of the universal enveloping algebras $U(\mathfrak{sl}_n)$ of the Lie algebras \mathfrak{sl}_n . Since these algebras are closely related to the Heisenberg algebra it is perhaps not surprising that such a treatment can be given. The treatments of both families of algebras is unified in the accompanying article [18], in which the representations of the categorified Heisenberg algebra and categorified $U(\mathfrak{sl}_n)$ are both seen as arising from free symmetric monoidal groupoids: for the Heisenberg algebra, on the trivial groupoid with one morphism, and for $U(\mathfrak{sl}_n)$, on the discrete groupoid with n morphisms. The construction relating Heisenberg algebras and $U(\mathfrak{sl}_n)$ is related to Kac-Moody algebras and their categorifications [21], so groupoidification may also provide a useful perspective in this case.

In the q -deformed case, categorifications in terms of 2-vector spaces have been described as a part of the Khovanov-Lauda programme [14]. The groupoidification formalism used here, based on the groupoid of finite sets and bijections, cannot be directly applied in this case. Baez, Hoffnung and Walker [2] have suggested that such groupoidifications should not be in terms of bijections of sets, but rather linear bijections of vector spaces over the finite field \mathbb{F}_q with q elements, for q a prime power. These would be used as groupoidifications of the Hecke algebras which appear in place of symmetric group algebras in the categorifications of quantum groups. If successful, this would yield combinatorial models for these categorified q -deformed algebras, yielding further insight into their structure.

Acknowledgements

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2 Groupoidification of the Heisenberg algebra

2.1 Spans of groupoids

Our combinatorial representation of the categorified Heisenberg algebra will be given in terms of *spans of groupoids*. A groupoid is a category in which all morphisms are invertible, and a span of groupoids from \mathbf{A} to \mathbf{B} is a diagram of the following form, where \mathbf{A} , \mathbf{B} and \mathbf{X} are groupoids and F and G are functors:

$$\begin{array}{ccc} & \mathbf{X} & \\ G \swarrow & & \searrow F \\ \mathbf{B} & & \mathbf{A} \end{array} \quad (9)$$

A span like this forms a 1-morphism of type $\mathbf{A} \rightarrow \mathbf{B}$ in the monoidal 2-category $\mathbf{Span}(\mathbf{Gpd})$. The symmetry in the definition of a span means that, for any span $\mathbf{B} \xleftarrow{G} \mathbf{X} \xrightarrow{F} \mathbf{A}$ as above, we can define its *converse* $(\mathbf{B} \xleftarrow{G} \mathbf{X} \xrightarrow{F} \mathbf{A})^\dagger$ as the span $\mathbf{A} \xleftarrow{F} \mathbf{X} \xrightarrow{G} \mathbf{B}$. We will note that this converse is in fact an adjoint.

If we think of the objects of \mathbf{A} and \mathbf{B} as being the states of some physical system, then any $x \in \text{Ob}(\mathbf{X})$ can be interpreted as a ‘history’ relating the state $F(x) \in \text{Ob}(\mathbf{A})$ to the state $G(x) \in \text{Ob}(\mathbf{B})$. Because we are working with groupoids, our states and histories come equipped with symmetry groups, and the functors F and G show how the symmetries of histories are mapped to symmetries of states.

The combinatorial interpretation that we will explore for these spans arises from the fundamental role played here by the groupoid of finite sets. The Fock space is represented by the groupoid \mathbf{S} and the annihilation and creation operators A and A^\dagger by the following spans of groupoids:

$$\begin{array}{ccc} & \mathbf{S} & \\ \text{id} \swarrow & & \searrow +1 \\ \mathbf{S} & & \mathbf{S} \end{array} \qquad \begin{array}{ccc} & \mathbf{S} & \\ +1 \swarrow & & \searrow \text{id} \\ \mathbf{S} & & \mathbf{S} \end{array} \quad (10)$$

Here \mathbf{S} is the groupoid of finite sets and bijections, and $\mathbf{S} \xrightarrow{+1} \mathbf{S}$ is the functor taking the disjoint union with the one-element set.

A different perspective on \mathbf{S} is to see it as the *free symmetric monoidal groupoid* on the trivial groupoid $\mathbf{1}$, with one object and one morphism. The functor $\mathbf{S} \xrightarrow{+1} \mathbf{S}$ then arises by taking the tensor product with the generating object. In fact, this free symmetric monoidal perspective is fundamental, and allows for the construction of a representation of the categorified Heisenberg algebra in completely abstract terms. We develop this perspective in detail in the companion article [18].

2.2 Combinatorics

Spans of groupoids can be used to reason about the combinatorics of structures on finite sets. The theory of *stuff types* [1] has been developed to describe how this works, generalizing Joyal’s theory of *structure types* [12]. Given some particular structure of interest constructed from a finite set, the groupoid \mathbf{G} of *models* of this structure can be built, with morphisms given by symmetries that relate one model to another. This comes equipped with a functor $\mathbf{G} \xrightarrow{F} \mathbf{S}$, called a *stuff type*, where \mathbf{S} is the groupoid of finite sets and bijections. This functor assigns to each model its underlying set, and to symmetries of models the induced bijections on the underlying sets. We can interpret any abstract stuff type as describing some combinatorial structure; in general, a finite set equipped with some structure which satisfies some property. A stuff type with good properties can be *degroupoidified*, a form of *linearization*, giving rise to a generating function for the structure in the ordinary sense.

Since every groupoid has a unique functor to the 1-element set, stuff types are precisely spans of the form

$$\begin{array}{ccc} & \mathbf{G} & \\ F \swarrow & & \searrow \\ \mathbf{S} & & \mathbf{1}, \end{array} \quad (11)$$

which are morphisms of type $\mathbf{1} \rightarrow \mathbf{S}$ in $\mathbf{Span}(\mathbf{Gpd})$. A *stuff operator* is a span of groupoids

of type $\mathbf{S} \rightarrow \mathbf{S}$:

$$\begin{array}{ccc}
 & \mathbf{H} & \\
 K \swarrow & & \searrow J \\
 \mathbf{S} & & \mathbf{S}
 \end{array} \tag{12}$$

A stuff operator thus contains a single groupoid of models, but two potentially different descriptions of underlying sets; it tells us how the same structure can be built in different ways. In the physical interpretation we have mentioned, these models — the objects in the middle groupoid — are seen as *histories*, and the two underlying sets are the starting and ending configurations of these histories. Stuff operators can act on stuff types by composition in $\mathbf{Span}(\mathbf{Gpd})$, producing new stuff types whose structures are composites of those from the original stuff type with the histories from the stuff operator.

For example, consider the stuff type $\mathbf{S} \xrightarrow{+1} \mathbf{S}$, which takes the union of each set with a chosen 1-element set. This represents the structure “finite sets with a chosen element”, with symmetries given by bijections that leave the chosen element fixed. Then our annihilation operator A defined in (10) is a stuff operator which treats an n -element set in the middle copy of \mathbf{S} in two different ways: by the right leg, as an $n+1$ -element set with a chosen element; and by the left leg, simply as an n -element set. It can be thought of as a ‘rule’ for how to consider an $n+1$ -element set as an n -element set — or more simply, as a way to *remove an element* from a finite set.

2.3 Composing spans

Given spans of groupoids $\mathbf{B} \xleftarrow{G} \mathbf{X} \xrightarrow{F} \mathbf{A}$ and $\mathbf{C} \xleftarrow{K} \mathbf{Y} \xrightarrow{J} \mathbf{B}$, we can compose them by constructing a *weak pullback* groupoid $(J \downarrow G)$:

$$\begin{array}{ccccc}
 & & (J \downarrow G) & & \\
 & \swarrow P_Y & & \searrow P_X & \\
 K \circ P_Y & & & & F \circ P_X \\
 \swarrow & & \alpha & & \searrow \\
 \mathbf{C} & \xleftarrow{K} \mathbf{Y} & \xrightarrow{J} \mathbf{B} & \xleftarrow{G} \mathbf{X} & \xrightarrow{F} \mathbf{A}
 \end{array} \tag{13}$$

This groupoid $(J \downarrow G)$ is equipped with functors $(J \downarrow G) \xrightarrow{P_Y} \mathbf{Y}$ and $(J \downarrow G) \xrightarrow{P_X} \mathbf{X}$, and a natural isomorphism $J \circ P_Y \xrightarrow{\alpha} G \circ P_X$. It is defined to satisfy a universal property: for any groupoid \mathbf{Z} equipped with functors $\mathbf{Z} \xrightarrow{Z_Y} \mathbf{Y}$ and $\mathbf{Z} \xrightarrow{Z_X} \mathbf{X}$ and a natural isomorphism $J \circ Z_X \xrightarrow{\zeta} G \circ Z_Y$, there must exist a functor $\mathbf{Z} \xrightarrow{L} (J \downarrow G)$, unique up to isomorphism, such that $L \circ \alpha = \zeta$. The resulting composite span is defined to be $\mathbf{C} \xleftarrow{K \circ P_Y} (J \downarrow G) \xrightarrow{F \circ P_X} \mathbf{A}$.

Based on the universal property described above, a standard construction for $(J \downarrow G)$ is the following groupoid:

- **Objects** are triples $(x \in \text{Ob}(\mathbf{X}), y \in \text{Ob}(\mathbf{Y}), G(x) \xrightarrow{f} J(y))$.
- **Morphisms** $(x_1, y_1, f_1) \rightarrow (x_2, y_2, f_2)$ are pairs of morphisms $x_1 \xrightarrow{a} x_2$ and $y_1 \xrightarrow{b} y_2$

satisfying the following commuting diagram:

$$\begin{array}{ccc}
G(x_1) & \xrightarrow{f_1} & J(y_1) \\
G(a) \downarrow & & \downarrow J(b) \\
G(x_2) & \xrightarrow{f_2} & J(y_2)
\end{array} \tag{14}$$

This construction is essentially a “weak” form of the fibred product, where instead of taking pairs (x, y) whose images in \mathbf{B} agree, we choose a specific isomorphism between them. We can unpack what this means for stuff types, for which $\mathbf{A} = \mathbf{B} = \mathbf{C} = \mathbf{S}$, in a concise way: an object in the weak pullback is a pair of models in \mathbf{X} and \mathbf{Y} equipped with a bijection of underlying sets; and a morphism is a pair of symmetries of models which induce the same bijections on the underlying sets.

2.4 The categorified commutation relation

The categorified form of the commutation relation (1) is an isomorphism of spans of the following form:

$$A \circ A^\dagger \simeq A^\dagger \circ A \oplus \text{id}_{\mathbf{S}} \tag{15}$$

The symbol ‘ \oplus ’ represents the *direct sum* of spans, which is just the disjoint union of groupoids of histories.

We begin with an intuitive argument for why this isomorphism should exist. We saw above that the histories of the span A^\dagger represent all the ways to add an element to a finite set, and the histories of A represent all the ways to remove an element. The histories of $A \circ A^\dagger$ therefore represent all the ways to add an element x , and then remove an element y . Such histories can be divided into two distinct classes: those for which $x \neq y$, and those for which $x = y$. Restricting to the first case, one might as well remove y before adding x , and so we have a bijection to the histories for the span $A^\dagger \circ A$; in the second case, the set remains unchanged, corresponding to the identity span. These two cases give the two terms on the right-hand side, and the explicit case-by-case bijection we have constructed gives the isomorphism of spans.

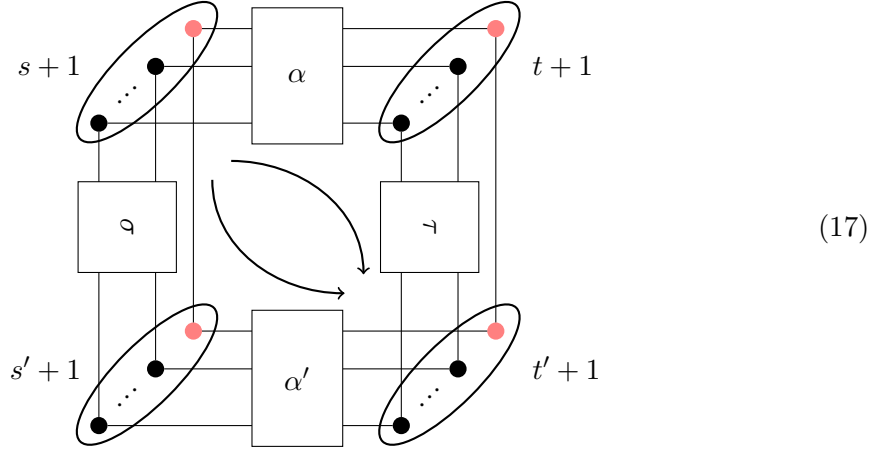
We now verify this intuition with direct calculation. The composite $A \circ A^\dagger$ is the following span:

$$\begin{array}{ccc}
& (+1 \downarrow +1) & \\
P \swarrow & & \searrow Q \\
\mathbf{S} & & \mathbf{S}
\end{array} \tag{16}$$

The groupoid $(+1 \downarrow +1)$ has objects which are triples (s, t, α) , where s and t are sets and $s + 1 \xrightarrow{\alpha} t + 1$ is an isomorphism. The projection maps P and Q have the actions $P(s, t, \alpha) = s$ and $Q(s, t, \alpha) = t$ on objects.

A morphism $(s, t, \alpha) \rightarrow (s', t', \alpha')$ is a pair of isomorphisms $s \xrightarrow{\sigma} s'$ and $t \xrightarrow{\tau} t'$ such that $\alpha' \circ (\sigma + 1) = (\tau + 1) \circ \alpha$, with the projection maps P and Q taking this to σ and τ

respectively. We can visualize this condition in the following way:



The extra element in each set is drawn in red. We show explicitly in this diagram how the permutations α , α' , σ and τ act on the different parts of each set. For the condition to be satisfied, the two composites from the top-left to the bottom-right must be equal.

With this diagram to aid us, we can draw some interesting conclusions. Firstly, for (s, t, α) and (s', t', α') to be isomorphic, then α must fix the extra element iff α' does — and this is sufficient as long as $s \simeq s'$ and $t \simeq t'$. So for each isomorphism class of object in its preimage under P or Q in $(+1 \downarrow +1)$. Secondly, if α and α' fix the extra element, then for every σ there exists some τ making the diagram commute, so there is an S_n -worth of morphisms between any two such objects for $n = |S|$. Thirdly, if neither α nor α' fix the extra element, then a given τ can only be part of a commuting diagram if $\tau(\alpha(1)) = \alpha'(1)$; in this case we can define α as the restriction to S of the composite $\alpha'^{-1} \circ (\tau + 1) \circ \alpha$. This means that, in this case, there is a S_{n-1} worth of morphisms between any two such objects for $n = |S|$.

This gives us a complete understanding of the isomorphism classes of object in $(+1 \downarrow +1)$ and their symmetries, which are the union of the entries in the following table:

	0	1	2	3	...	n	...
Extra element not fixed		S_0	S_1	S_2	...	S_{n-1}	...
Extra element fixed	S_0	S_1	S_2	S_3	...	S_n	...

The column headings indicate the cardinality of the set to which each object is projected, under the projection maps P and Q . As a result, we see that the span (16) can be written in the following way:

$$\begin{array}{ccc}
 & \mathbf{S} \cup \mathbf{S} & \\
 (\text{id}_{\mathbf{S}}, +1) \swarrow & & \searrow (\text{id}_{\mathbf{S}}, +1) \\
 \mathbf{S} & & \mathbf{S}
 \end{array} \tag{19}$$

Here $\mathbf{S} \cup \mathbf{S}$ represents the disjoint union of two copies of \mathbf{S} , and $(\text{id}_{\mathbf{S}}, +1)$ represents the functor which acts as the identity on the first copy of \mathbf{S} and as $+1$ on the second. We can consider this to be the union of spans $(\mathbf{S} \xleftarrow{\text{id}_{\mathbf{S}}} \mathbf{S} \xrightarrow{\text{id}_{\mathbf{S}}} \mathbf{S})$ and $(\mathbf{S} \xleftarrow{+1} \mathbf{S} \xrightarrow{+1} \mathbf{S})$, which gives us the isomorphism (15) we are seeking.

2.5 Forming a 2-category

So far our construction essentially follows that given in [19], but we now need to go further and use the 2-categorical structure of $\mathbf{Span}(\mathbf{Gpd})$. This requires a notion of morphism between spans. For this, we define a *span of spans* of type $(\mathbf{B} \xleftarrow{G} \mathbf{X} \xrightarrow{F} \mathbf{A}) \rightarrow (\mathbf{B} \xleftarrow{J} \mathbf{Y} \xrightarrow{K} \mathbf{A})$ is a span $\mathbf{X} \xleftarrow{S} \mathbf{Z} \xrightarrow{T} \mathbf{Y}$ equipped with natural transformations $G \circ S \xrightarrow{\mu} J \circ T$ and $F \circ S \xrightarrow{\nu} K \circ T$, as indicated by the following diagram:

$$\begin{array}{ccccc}
 & & \mathbf{X} & & \\
 & G \swarrow & \uparrow S & \searrow F & \\
 \mathbf{B} & & \mathbf{Z} & & \mathbf{A} \\
 & \downarrow \mu & & \downarrow \nu & \\
 & J \swarrow & \downarrow T & \searrow K & \\
 & & \mathbf{Y} & &
 \end{array} \tag{20}$$

We say that two such spans of spans $(\mathbf{X} \xleftarrow{S} \mathbf{Z} \xrightarrow{T} \mathbf{Y}, \mu, \nu)$ and $(\mathbf{X} \xleftarrow{S'} \mathbf{Z}' \xrightarrow{T'} \mathbf{Y}, \mu', \nu')$ are *equivalent* when there is an equivalence of groupoids $\mathbf{Z} \xrightarrow{U} \mathbf{Z}'$, and natural transformations $S \xrightarrow{\sigma} S' \circ U$ and $T \xrightarrow{\tau} T' \circ U$, such that the following pasted composites give μ and ν respectively:

$$\begin{array}{ccc}
 \begin{array}{ccccc}
 & & \mathbf{X} & & \\
 & G \swarrow & \uparrow S & \searrow F & \\
 \mathbf{B} & & \mathbf{Z} & & \mathbf{A} \\
 & \downarrow \mu' & \uparrow U & \downarrow \nu' & \\
 & J \swarrow & \downarrow T & \searrow K & \\
 & & \mathbf{Y} & &
 \end{array} & \begin{array}{ccccc}
 & & \mathbf{X} & & \\
 & G \swarrow & \uparrow S' & \searrow F & \\
 \mathbf{B} & & \mathbf{Z}' & & \mathbf{A} \\
 & \downarrow \mu' & \uparrow U & \downarrow \nu' & \\
 & J \swarrow & \downarrow T' & \searrow K & \\
 & & \mathbf{Y} & &
 \end{array} & \\
 \end{array} \tag{21}$$

We use this to form a 2-category $\mathbf{Span}(\mathbf{Gpd})$ of spans of groupoids:

- **Objects** are groupoids.
- **Morphisms** are spans of groupoids, with composition defined by weak pullback.
- **2-Morphisms** are equivalence classes of spans of spans.

This 2-category will be the formal setting for our results. The symmetric monoidal 2-category structure of $\mathbf{Span}(\mathbf{Gpd})$ has been described by Hoffnung [11], although in the simpler situation where which the 2-morphisms are span maps rather than spans of spans. It is mechanical but tedious to describe the how the monoidal 2-category structure extends to our case.

The 2-category $\mathbf{Span}(\mathbf{Gpd})$ has strong duality properties. We have already defined a notion of converse for 1-morphisms. In addition, for any 2-morphism

$$(\mathbf{B} \xleftarrow{G} \mathbf{X} \xrightarrow{F} \mathbf{A}) \xrightarrow{(\mathbf{X} \xleftarrow{S} \mathbf{Z} \xrightarrow{T} \mathbf{Y}, \mu, \nu)} (\mathbf{B} \xleftarrow{J} \mathbf{Y} \xrightarrow{K} \mathbf{A}),$$

we can define its *converse* $(\mathbf{X} \xleftarrow{S} \mathbf{Z} \xrightarrow{T} \mathbf{Y}, \mu, \nu)^\dagger$ as the following span of spans:

$$(\mathbf{B} \xleftarrow{J} \mathbf{Y} \xrightarrow{K} \mathbf{A}) \xrightarrow{(\mathbf{Y} \xleftarrow{T} \mathbf{Z} \xrightarrow{S} \mathbf{X}, \mu^{-1}, \nu^{-1})} (\mathbf{B} \xleftarrow{G} \mathbf{X} \xrightarrow{F} \mathbf{A}).$$

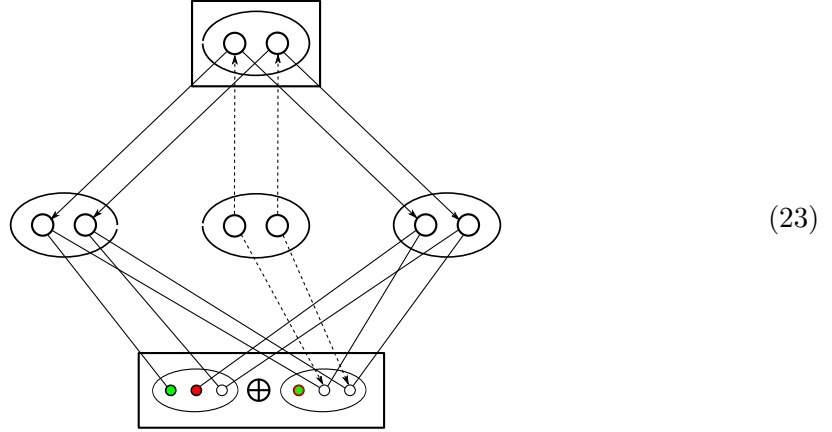
2.6 The commutation relation

Based on the calculations in the previous section, we can define the following spans of spans, called $i_{\text{id}_{\mathbf{S}}}$ and $i_{A \circ A^\dagger}$:

$$\begin{array}{ccc}
 \begin{array}{c}
 \text{S} \\
 \swarrow \text{id}_{\text{S}} \quad \searrow \text{id}_{\text{S}} \\
 \text{S} \quad \text{S} \\
 \downarrow \text{id} \quad \downarrow \text{id} \\
 \text{S} \quad \text{S} \\
 \swarrow (\text{id}_{\text{S}}, +1) \quad \searrow (\text{id}_{\text{S}}, +1) \\
 \text{S} \cup \text{S} \\
 \text{id}_{\text{S}} \xrightarrow{i_{\text{id}_{\text{S}}}} A \circ A^\dagger
 \end{array}
 &
 \begin{array}{c}
 \text{S} \\
 \swarrow +1 \quad \searrow +1 \\
 \text{S} \quad \text{S} \\
 \downarrow \text{id} \quad \downarrow \text{id} \\
 \text{S} \quad \text{S} \\
 \swarrow (\text{id}_{\text{S}}, +1) \quad \searrow (\text{id}_{\text{S}}, +1) \\
 \text{S} \cup \text{S} \\
 A^\dagger \circ A \xrightarrow{i_{A^\dagger \circ A}} A \circ A^\dagger
 \end{array}
 & (22)
 \end{array}$$

Here, I_1 and I_2 are functors embedding the first and second copy of \mathbf{S} respectively.

We look more closely at the definition of $i_{\text{id}_{\mathbf{S}}}$, looking ‘inside’ the groupoids for the case of histories acting on the 2-element set:



Across the top of this picture, we see one object of the middle \mathbf{S} , namely the two-element set. The identity span maps this object down to the two-element set on either side. On the bottom, though, we see the two objects in $(+1 \downarrow +1)$ which map down to two-element sets. Each is a 3-element set, but as in the table (18), we see that the marked elements — the element added and the element removed — are either the same or different. The symmetry groups noted in (18) are those which permute the unmarked elements. The span of spans shows how objects in the identity relate to particular objects in $(+1 \downarrow +1)$. This determines an inclusion.

We can now give a formal interpretation of the commutation relation

$$A \circ A^\dagger \simeq A^\dagger \circ A \oplus \text{id}_{\mathbf{S}}. \quad (24)$$

We interpret the \oplus symbol as a \dagger -biproduct, a categorical generalization of a direct sum. Such a biproduct is witnessed by the following equations involving the injection 2-mor-

phisms $i_{\text{id}_{\mathbf{S}}}$, $i_{A^\dagger \circ A}$ and their converses:

$$\text{id}_{\text{id}_{\mathbf{S}}} = (i_{\text{id}_{\mathbf{S}}})^\dagger \circ i_{\text{id}_{\mathbf{S}}} \quad (25)$$

$$0_{A^\dagger \circ A, \text{id}_{\mathbf{S}}} = (i_{\text{id}_{\mathbf{S}}})^\dagger \circ i_{A^\dagger \circ A} \quad (26)$$

$$0_{\text{id}_{\mathbf{S}}, A^\dagger \circ A} = (i_{A^\dagger \circ A})^\dagger \circ i_{\text{id}_{\mathbf{S}}} \quad (27)$$

$$\text{id}_{A^\dagger \circ A} = (i_{A^\dagger \circ A})^\dagger \circ i_{A^\dagger \circ A} \quad (28)$$

$$\text{id}_{A \circ A^\dagger} = i_{\text{id}_{\mathbf{S}}} \circ (i_{\text{id}_{\mathbf{S}}})^\dagger + i_{A^\dagger \circ A} \circ (i_{A^\dagger \circ A})^\dagger \quad (29)$$

These can be verified by direct calculation. We will see that these injection and projection maps, which characterize the \dagger -biproduct, are related to the adjointness of the spans A and A^\dagger .

2.7 Graphical notation

We now introduce a graphical notation for certain spans of type $\mathbf{S} \leftarrow \mathbf{X} \rightarrow \mathbf{S}$, and for the 2-morphisms going between them in $\mathbf{Span}(\mathbf{Gpd})$. This is the notation of Khovanov [13], and is an application of the standard graphical calculus for morphisms in a monoidal category [23]. Creation operators A^\dagger and annihilation operators A are represented as vertical lines with upwards and downwards orientation, respectively:

$$\begin{array}{c} \downarrow \\ \uparrow \\ A^\dagger \end{array} \quad \begin{array}{c} \downarrow \\ A \end{array} \quad (30)$$

Composition of operators is represented by horizontal juxtaposition. The identity span $\mathbf{S} \xleftarrow{\text{id}_{\mathbf{S}}} \mathbf{S} \xrightarrow{\text{id}_{\mathbf{S}}} \mathbf{S}$ is represented by the empty diagram.

We denote 2-morphisms with string diagrams. In particular, our 2-morphisms $i_{\text{id}_{\mathbf{S}}}$, $(i_{\text{id}_{\mathbf{S}}})^\dagger$, $i_{A^\dagger \circ A}$ and $(i_{A^\dagger \circ A})^\dagger$ have the following representations:

$$\begin{array}{cccc} \begin{array}{c} \downarrow \\ \uparrow \end{array} & \begin{array}{c} \downarrow \\ \uparrow \end{array} & \begin{array}{c} \downarrow \\ \uparrow \end{array} & \begin{array}{c} \downarrow \\ \uparrow \end{array} \\ i_{\text{id}_{\mathbf{S}}} & (i_{\text{id}_{\mathbf{S}}})^\dagger & i_{A^\dagger \circ A} & (i_{A^\dagger \circ A})^\dagger \end{array} \quad (31)$$

Using this notation, the biproduct equations (25–29) have the following representation:

$$\begin{array}{ccc} \begin{array}{c} \downarrow \\ \uparrow \end{array} = \text{id}_{\text{id}_{\mathbf{S}}} & \begin{array}{c} \downarrow \\ \uparrow \end{array} = 0_{A^\dagger \circ A, \text{id}_{\mathbf{S}}} & \begin{array}{c} \downarrow \\ \uparrow \end{array} = 0_{\text{id}_{\mathbf{S}}, A^\dagger \circ A} \\ (25) & (26) & (27) \end{array}$$

$$\begin{array}{ccc} \begin{array}{c} \downarrow \\ \uparrow \end{array} = \begin{array}{c} \downarrow \\ \uparrow \end{array} & + & \begin{array}{c} \downarrow \\ \uparrow \end{array} = \begin{array}{c} \downarrow \\ \uparrow \end{array} \\ (28) & & (29) \end{array}$$

Khovanov's graphical axioms for the categorified Heisenberg algebra include equations (25),

(28) and (29). Equations (26) and (27) are not explicitly part of his algebra, but they can be straightforwardly derived from the other three in a setting where hom-set addition distributes over composition and addition is cancellable, properties which hold in both $\mathbf{Span}(\mathbf{Gpd})$ and also in Khovanov's bimodule category setting.

Although the graphical representations of $i_{A^\dagger \circ A}$ and its converse look like braiding of a braided monoidal category, we do not have such a structure here, as witnessed explicitly by equation (29): the 'braidings' are not invertible. It even fails to be a lax braiding, as naturality fails to hold. We will see in Section 2.11 how to deduce this from a theorem of Yetter [28].

2.8 Combinatorial interpretation

We can extend our combinatorial interpretation to the morphisms $i_{\mathbf{id}_S}$, $(i_{\mathbf{id}_S})^\dagger$, $i_{A^\dagger \circ A}$ and $(i_{A^\dagger \circ A})^\dagger$ as depicted in expression (31). This will allow us to see intuitively why equations (25–29) should hold.

A span of spans gives a way to relate one history to another, in such a way that related histories have isomorphic sources and isomorphic targets. We can therefore understand our spans of spans by listing the histories which they relate. This does not completely define a span of spans, as the actions on the symmetry groups of objects must also be taken into account, but it is nevertheless a useful way to develop intuition.

- $\mathbf{id}_S \xrightarrow{i_{\mathbf{id}_S}} A \circ A^\dagger$. This relates a history where no change is made to the underlying set, to a history in which some element is added and then removed.
- $A \circ A^\dagger \xrightarrow{(i_{\mathbf{id}_S})^\dagger} \mathbf{id}_S$. Histories in $A \circ A^\dagger$ representing adding and removing the same element are related to the trivial history representing no action. Histories representing adding one element and removing a different element are related to nothing.
- $A^\dagger \circ A \xrightarrow{i_{A^\dagger \circ A}} A \circ A^\dagger$. Histories where x is removed and then y is added are related to histories where y is added and then x is removed.
- $A \circ A^\dagger \xrightarrow{(i_{A^\dagger \circ A})^\dagger} A^\dagger \circ A$. Given a history where y is added and x is removed, then if $x \neq y$ this is related to the history where x is removed and then y is added; otherwise it is related to nothing.

This intuition allows us to understand why equations (25–29) should hold.

- (25): If we begin with the trivial action on a set, and then pass to a history where we add and remove the same element, and then verify that indeed the same element has been added as has been removed, then this leaves our initial history unchanged.
- (26): Beginning with a history where we remove x and add y , we then ensure that $x \neq y$ and pass to a history where we add y and then remove x . We then ensure that x and y are the same, which is clearly impossible, so our original history is related to nothing.
- (27): We begin by choosing a history where we add and remove the same element. We then reverse the order of these operations, which is clearly impossible, so our original history is related to nothing.
- (28): We begin with a history where we remove x and add y . This is related to a history where we add y and then remove x , which in turn is related to a history where we remove x and add y . This clearly gives the identity on histories.

- (29): The left-hand side of this equation is comprised of two separate relations on histories. Suppose we begin with a history for which we add x and then remove y . The first summand selects the case that $x = y$, and relates it to itself. The second ensures that $x \neq y$ and relates the initial history to the case that we remove y and then add x , which in turn is related to the history where we add x and then remove y ; so this relates every history to itself, except for the case that $x = y$. Overall the sum of these relations give the identity on histories, and so the equation is satisfied.

2.9 Adjunction of A and A^\dagger

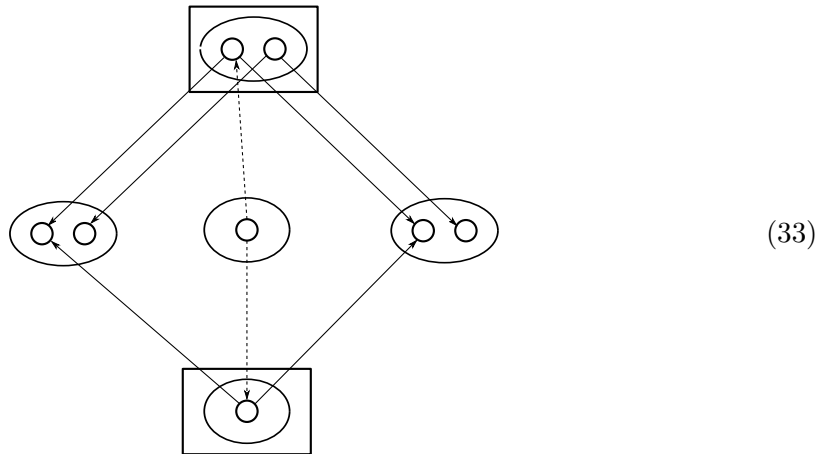
The 2-category $\mathbf{Span}(\mathbf{Gpd})$ is useful because it contains \mathbf{Gpd} , and every morphism has a two-sided adjoint, which is a feature of the span construction [4]. In particular, for every span, its converse is both a left and a right adjoint. This situation is called an *ambidextrous adjunction*, or just *ambijunction* for short. (The importance of such a relation in a model of diagrammatic categorification is a result of the fact that a certain operations in a diagrammatic calculus express the relation of ambijunction graphically [15].)

To see how this arises for our annihilation and creation operators A and A^\dagger , we introduce the following span of spans named η :

$$\begin{array}{ccccc}
 & & \mathbf{S} & & \\
 & \swarrow \text{id}_{\mathbf{S}} & \uparrow +1 & \searrow \text{id}_{\mathbf{S}} & \\
 \mathbf{S} & & \mathbf{S} & & \mathbf{S} \\
 & \swarrow +1 & \downarrow \text{id} & \searrow +1 & \\
 & & \mathbf{S} & & \\
 & & \downarrow \text{id}_{\mathbf{S}} & & \\
 & & \mathbf{S} & &
 \end{array} \quad (32)$$

$$\text{id}_{\mathbf{S}} \xrightarrow{\eta} A^\dagger \circ A$$

We can understand how this acts with the following diagram, which depicts the action of η at the 2-element set. Of course, a similar picture could be drawn for each nonempty set in \mathbf{S} .



We can understand η and its converse combinatorially in the following way.

- $\text{id}_{\mathbf{S}} \xrightarrow{\eta} A^\dagger \circ A$. This relates the identity history to the history where an element is removed, and then added. This is impossible on the zero-element set, and so in that case η relates the identity history to nothing.

- $A^\dagger \circ A \xrightarrow{\eta^\dagger} \text{id}_S$. This relates any history to the identity history.

This extends our interpretation for other structures given in Section 2.8.

The graphical representation is extended to η and its converse in the following way:

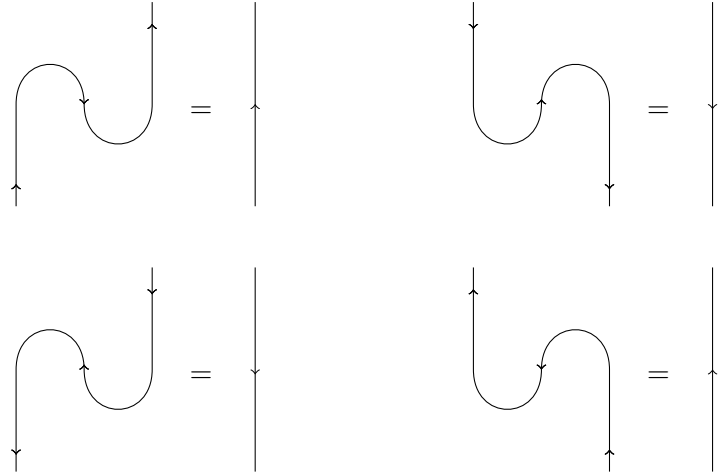


$$\eta \quad (34)$$



$$\eta^\dagger \quad (35)$$

Along with i_{id_S} and $(i_{\text{id}_S})^\dagger$, these 2-morphisms satisfy adjunction equations witnessing both $A \dashv A^\dagger$ and $A^\dagger \dashv A$. These equations have the following representation in our graphical notation:




$$(36)$$

We can interpret these snake equations combinatorially in a similar way to equations (25–29). We examine the first in detail. On the left-hand side, we begin with the operator A^\dagger , a history of which corresponds to adding some element x to a set. We then apply i_{id_S} , passing to a history for which we add some element y , remove it, and then add our element x . Finally we apply our interpretation of η^\dagger , giving a history where we add the element y . This is not the same as our original history, for which we added the element x , but it is *equivalent* to it, which is our definition of equality for 2-morphisms in $\mathbf{Span}(\mathbf{Gpd})$ as given by equation (21). The other snake equations can be interpreted in a similar way.

2.10 Symmetric group actions

The construction of the full categorified Heisenberg algebra makes use of certain “symmetrizer” objects. This relies on the fact that on spans of the form A^n or $(A^\dagger)^n$ we have an action of the symmetric group S_n . The effect of this action is to permute the order in which elements of a set are added or removed. Up to isomorphism, spans of the form A^n and $(A^\dagger)^n$ have the following form:



$$(37)$$

Here we define $(+n) := (+1)^n$, the functor which takes the disjoint union with an n -element set. It has natural endo-transformations $\hat{\mu}$ for each $\mu \in S_n$, defined for each set

T by the set maps

$$\hat{\mu}_T : T \sqcup \mathbf{n} \rightarrow T \sqcup \mathbf{n}, \quad (38)$$

which act by $\hat{\mu}_T(t) = t$ for all $t \in T$ and $\hat{\mu}_T(j) = \mu(j)$ for all $j \in \mathbf{n}$. These give rise to the following spans of spans:

$$\begin{array}{ccc} & \mathbf{S} & \\ \text{id}_{\mathbf{S}} \swarrow & \uparrow \text{id}_{\mathbf{S}} & \searrow +n \\ \mathbf{S} & \Downarrow \text{id} & \mathbf{S} \\ \text{id}_{\mathbf{S}} \swarrow & \uparrow \text{id}_{\mathbf{S}} & \searrow +n \\ & \mathbf{S} & \end{array} \quad \begin{array}{ccc} & \mathbf{S} & \\ +n \swarrow & \uparrow \text{id}_{\mathbf{S}} & \searrow \text{id}_{\mathbf{S}} \\ \mathbf{S} & \Downarrow \hat{\mu} & \mathbf{S} \\ +n \swarrow & \uparrow \text{id}_{\mathbf{S}} & \searrow \text{id}_{\mathbf{S}} \\ & \mathbf{S} & \end{array} \quad (39)$$

For the action of the nonidentity element of S_2 on A^2 and $(A^\dagger)^2$, we use the following graphical representation:

$$\begin{array}{c} \downarrow \quad \downarrow \\ \diagdown \quad \diagup \\ \uparrow \quad \uparrow \end{array} \quad \begin{array}{c} \downarrow \quad \downarrow \\ \diagup \quad \diagdown \\ \uparrow \quad \uparrow \end{array} \quad (40)$$

Actions of S_n on A^n and $(A^\dagger)^n$ can be built up from these basic permutations of adjacent operations. Because they are actions of the symmetric group, they satisfy the following equations:

$$\begin{array}{c} \downarrow \quad \downarrow \\ \diagdown \quad \diagup \\ \uparrow \quad \uparrow \end{array} = \begin{array}{c} \downarrow \quad \downarrow \\ | \quad | \\ \uparrow \quad \uparrow \end{array} \quad \begin{array}{c} \downarrow \quad \downarrow \\ \diagup \quad \diagdown \\ \uparrow \quad \uparrow \end{array} = \begin{array}{c} \downarrow \quad \downarrow \\ | \quad | \\ \uparrow \quad \uparrow \end{array} \quad (41)$$

They also satisfy the following equations, making them generators of S_n for any n :

$$\begin{array}{c} \downarrow \quad \downarrow \\ \diagdown \quad \diagup \\ \uparrow \quad \uparrow \end{array} = \begin{array}{c} \downarrow \quad \downarrow \\ \diagup \quad \diagdown \\ \uparrow \quad \uparrow \end{array} \quad \begin{array}{c} \downarrow \quad \downarrow \\ \diagup \quad \diagdown \\ \uparrow \quad \uparrow \end{array} = \begin{array}{c} \downarrow \quad \downarrow \\ \diagdown \quad \diagup \\ \uparrow \quad \uparrow \end{array} \quad (42)$$

As relations on histories, the symmetry on $A \circ A$ relates the history “remove x , and then remove y ” to the history “remove y , and then remove x ”. The symmetry on $A^\dagger \circ A^\dagger$ can be described in a similar way. The equations satisfied by these symmetries can then be accounted for using these combinatorial interpretations.

Together with the morphism $i_{A^\dagger \circ A}$ and its converse, these morphisms allow us to interpret arbitrary braid diagrams involving strands labelled A or A^\dagger . These are not invertible in general, since $i_{A^\dagger \circ A}$ is not an isomorphism but an inclusion, and so we do not get a representation of the symmetric group in these cases where both A and A^\dagger appear.

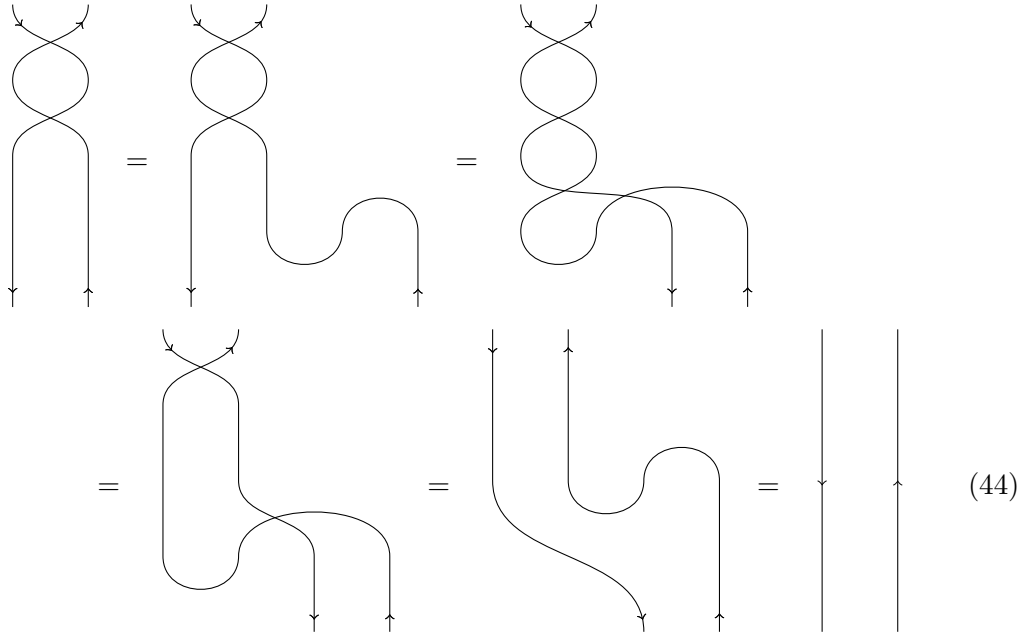
This is important for connections to quantum field theory, for which powers of the field operator $\Phi = A + A^\dagger$ are important. Thus, the inner product $\langle \psi, p(\Phi)\psi' \rangle$ (for a polynomial p) is the groupoid cardinality of the stuff type inner product

$$\langle \Psi, p(\Phi)\Psi' \rangle \quad (43)$$

where Ψ, Ψ' are any groupoidifications of the vectors ψ, ψ' . As described in [19], this can be interpreted as a *sum over histories*, which are represented by Feynman diagrams. These are given weights which are exactly as determined by the groupoid cardinality.

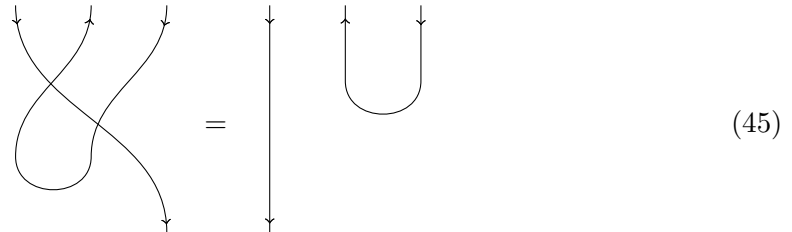
2.11 Failure of naturality

By a theorem of Yetter [28], a lax braiding for an object with a right dual is automatically invertible. Since our ‘braidings’ are not invertible, one of the axioms of a lax braiding must fail, and the culprit is naturality. If we *assume* naturality, we can use Yetter’s argument to show that the braiding is invertible, in contradiction with equation (29):



(44)

Combinatorially, the following naturality property fails to hold:

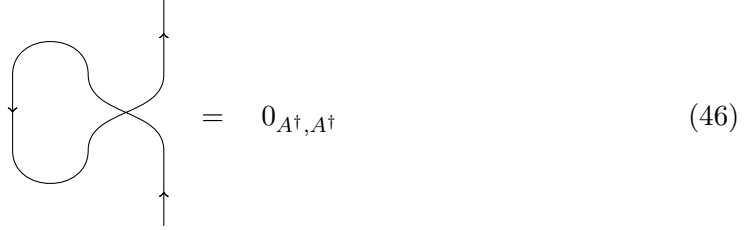


(45)

The reason this fails is that the left-hand side factors through the composite $A^\dagger \circ A \circ A$, and hence annihilates the history that removes the unique element from a 1-element set. The right-hand does not annihilate this history, and hence the two spans of spans cannot be equal.

2.12 The twist

A final axiom of Khovanov's has the following representation.



$$= 0_{A^\dagger, A^\dagger} \quad (46)$$

We interpret this combinatorially as follows. The initial span is A^\dagger , and we begin with a history in this span which adds an element x to a set. This is related to a history in which we add x , then add and remove some new element y , according to our interpretation of i_{id_S} as described above. We then apply the symmetry map, obtaining a history in which we add y , add x , and then remove y . Finally we apply our interpretation of $(i_{\text{id}_S})^\dagger$ to the final add-remove pair, which only relates histories in which the element we remove is the same as the element which we add. However, in this case this is impossible, as x and y are different elements, and so our history is related to nothing. Combinatorially, this says that we are distinguishing x and y .

3 Linearizing the groupoid representation

3.1 Introduction

Linearization is the process of turning geometrical structures into linear ones. An important form of linearization is *degrouoidification*, first described in [1] and surveyed in [2], which is a monoidal functor

$$\mathbf{Span}(\mathbf{Gpd})_1 \xrightarrow{D} \mathbf{Vect},$$

where $\mathbf{Span}(\mathbf{Gpd})_1$ is the monoidal 1-category obtained by identifying isomorphic 1-cells in the monoidal 2-category $\mathbf{Span}(\mathbf{Gpd})$. This functor D takes groupoids to the free vector space on their set of equivalence classes of objects, and spans of groupoids to certain linear maps between these.

We will use a higher-categorical generalization of this called *2-linearization*, which is a monoidal 2-functor

$$\mathbf{Span}(\mathbf{Gpd}) \xrightarrow{\Delta} \mathbf{2Vect}, \quad (47)$$

where $\mathbf{2Vect}$ is the monoidal 2-category of *Kapranov-Voevodsky 2-vector spaces*. This has objects given by \mathbb{C} -linear semisimple additive categories, 1-morphisms given by linear functors, and 2-morphisms given by natural transformations.

In Section 1.2 we described Khovanov's construction of the categorified Heisenberg algebra as a linear monoidal category \mathbf{H}' . In Section 2, we described how this acts in a natural way by spans and spans-of-spans over the groupoid \mathbf{S} of finite sets and bijections, yielding a combinatorial interpretation. This gives us a functor

$$\mathbf{H}' \xrightarrow{C} \mathbf{Span}(\mathbf{Gpd}), \quad (48)$$

where we see \mathbf{H}' as a bicategory with one object, which is sent by C to the object \mathbf{S} in $\mathbf{Span}(\mathbf{Gpd})$. We can think of this as a *representation* of \mathbf{H}' , in the same way that we can encode a group as a one-object category \mathbf{G} and think of a representation as a functor $\mathbf{G} \rightarrow \mathbf{Vect}$.

Composing the functors C and Λ , we obtain a new representation of \mathbf{H}' in $\mathbf{2Vect}$:

$$\mathbf{H}' \xrightarrow{C} \mathbf{Span}(\mathbf{Gpd}) \xrightarrow{\Lambda} \mathbf{2Vect} \quad (49)$$

As we shall see, this amounts to the representation that Khovanov describes in [13], in terms of the representation categories of the symmetric groups and bimodules between the group algebras. The factorization above shows that this representation of the diagrammatic category \mathbf{H}' can be seen to arise from the combinatorial interpretation of \mathbf{H}' in $\mathbf{Span}(\mathbf{Gpd})$.

3.2 Linearization

The 2-functor Λ is described in more detail in [20], but the essential starting point is that Λ takes a groupoid \mathbf{A} to its category $\text{Rep}(\mathbf{A})$ of finite-dimensional representations. A span of groupoids $\mathbf{B} \xleftarrow{G} \mathbf{X} \xrightarrow{F} \mathbf{A}$ is mapped to the composite functor

$$\text{Rep}(\mathbf{A}) \xrightarrow{F_*} \text{Rep}(\mathbf{X}) \xrightarrow{G^*} \text{Rep}(\mathbf{B}), \quad (50)$$

where F_* is the pullback functor for F , and G^* is the adjoint of the pullback functor for G . For a general span-of-spans of the form

$$(\mathbf{B} \xleftarrow{G} \mathbf{X} \xrightarrow{F} \mathbf{A}) \xrightarrow{(\mathbf{X} \xleftarrow{S} \mathbf{Z} \xrightarrow{T} \mathbf{Y}, \mu, \nu)} (\mathbf{B} \xleftarrow{J} \mathbf{Y} \xrightarrow{K} \mathbf{A}),$$

we construct its image under Λ as the following composite natural transformation:

$$\begin{array}{ccccc} & & \text{Rep}(\mathbf{X}) & & \\ & \swarrow G^* & \uparrow S^* \downarrow S_* & \nwarrow F_* & \\ \text{Rep}(\mathbf{B}) & & \text{Rep}(\mathbf{Z}) & & \text{Rep}(\mathbf{A}) \\ & \searrow J^* & \downarrow (\mu^*)^{-1} & \swarrow \nu_* & \\ & & \text{Rep}(\mathbf{Y}) & & \end{array} \quad (51)$$

$\begin{array}{c} \downarrow \nu_* \\ \text{Rep}(\mathbf{A}) \end{array} \xrightarrow{K_*} \text{Rep}(\mathbf{Y}) \xleftarrow{T^*} \text{Rep}(\mathbf{Z}) \xleftarrow{J^*} \text{Rep}(\mathbf{B})$

The central 2-cells arise from adjunctions $S_* \dashv S^*$ and $T^* \dashv T_*$.

3.3 2-Linearized ladder operators

To see why the composite (49) is isomorphic to Khovanov's representation, we consider the actions of the lowering and raising operators A and A^\dagger under the image of Λ . To begin with, the 2-vector space analog of 'Fock space' is:

$$\Lambda(\mathbf{S}) = [\mathbf{S}, \mathbf{Vect}] \simeq \coprod_{n \in \mathbb{N}} \text{Rep}(S_n) \quad (52)$$

This is a category generated by the irreducible representations of all the symmetric groups S_n . The categorified Heisenberg algebra acts on $\Lambda(\mathbf{S})$ by endofunctors and natural transformations, and we would like to understand this action concretely.

For each isomorphism class of object $n \in \mathbf{S}$, the irreducible representations of $\text{Aut}(n) = S_n$ are indexed by Young diagrams with n boxes, so there is a countable basis for $\Lambda(\mathbf{S})$

consisting of all Young diagrams. See [9] for the mechanics of Young diagrams in representation theory the symmetric groups, and [24] for a good discussion in a physical context.

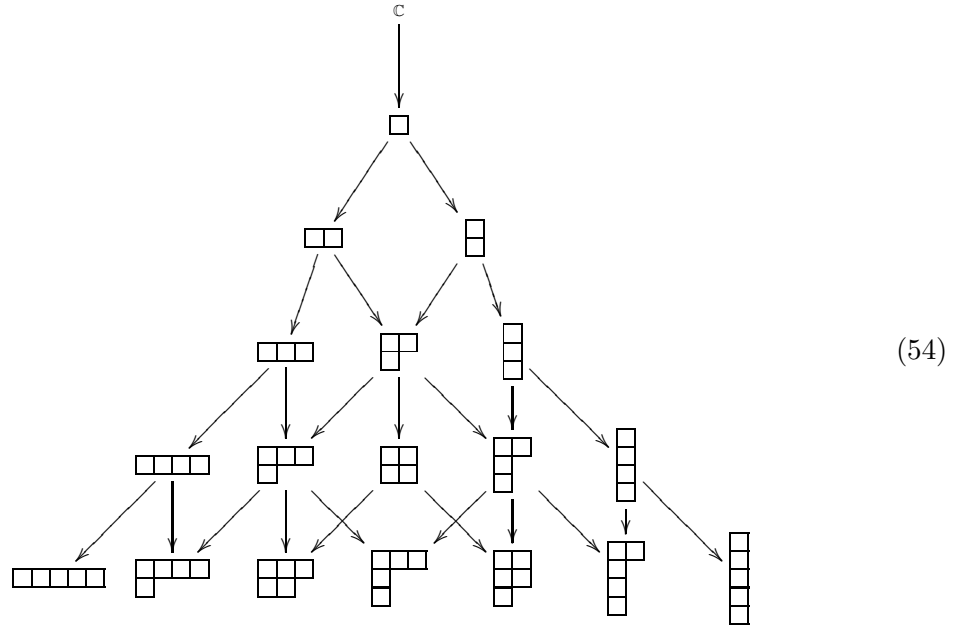
The 1-cell $\Lambda(A^\dagger)$ is given by the functor

$$\Lambda(\mathbf{S}) \xrightarrow{\Lambda(+1)} \Lambda(\mathbf{S}), \quad (53)$$

and $\Lambda(A)$ by its adjoint. For any n -element set, the functor $(+1)$ maps the automorphism group S_n into S_{n+1} , by the natural inclusion where S_n is the subgroup of S_{n+1} fixing the newly-added element.

To understand $\Lambda(A^\dagger)$ we must therefore see how a representation of \mathbf{S} is pulled back by the functor $(+1)$, and to understand $\Lambda(A)$ we must see how the adjoint of this pullback operation acts. This is described by the theory of restricted and induced representations, a standard part of the representation theory of the symmetric groups [22]. Given an irreducible representation of S_{n+1} represented by a Young diagram D , the corresponding restricted representation of S_n will be a direct sum of irreducibles, which correspond to all the valid Young diagrams obtained by deleting a single box from some row of D . Each representation corresponding to such a diagram appears in the restricted representation with multiplicity one. Likewise, given an irreducible representation of S_n represented by Young diagram D' , the induced representation of S_{n+1} is a representation in which every Young diagram arising by adding a box to any row of D' (possibly of zero length) appears with multiplicity one.

We can see the effect of $\Lambda(A)$ and $\Lambda(A^\dagger)$ on small representations of small symmetric groups using the following directed graph of Young diagrams:



Given any diagram in this partial order, the effect of $\Lambda(A)$ is to give the direct sum of every diagram immediately above it, and the effect of $\Lambda(A^\dagger)$ is to give the direct sum of every diagram immediately below it. As a result, the effect of multiple applications of $\Lambda(A)$ and $\Lambda(A^\dagger)$ can be calculated by counting paths.

We can represent this graph in the following way:

$$\Lambda(A) = \begin{pmatrix} 0 & M_{0,1} & 0 & 0 & 0 & 0 & \dots \\ 0 & 0 & M_{1,2} & 0 & 0 & 0 & \dots \\ 0 & 0 & 0 & M_{2,3} & 0 & 0 & \dots \\ 0 & 0 & 0 & 0 & M_{3,4} & 0 & \dots \\ 0 & 0 & 0 & 0 & 0 & M_{4,5} & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \quad (55)$$

Entries in this matrix are themselves matrices, the first nontrivial elements having the following forms:

$$M_{0,1} = (\mathbb{C}) \quad (56)$$

$$M_{1,2} = \begin{pmatrix} \mathbb{C} & \mathbb{C} \end{pmatrix} \quad (57)$$

$$M_{2,3} = \begin{pmatrix} \mathbb{C} & \mathbb{C} & 0 \\ 0 & \mathbb{C} & \mathbb{C} \end{pmatrix} \quad (58)$$

$$M_{3,4} = \begin{pmatrix} \mathbb{C} & \mathbb{C} & 0 & 0 & 0 \\ 0 & \mathbb{C} & \mathbb{C} & \mathbb{C} & 0 \\ 0 & 0 & 0 & \mathbb{C} & \mathbb{C} \end{pmatrix} \quad (59)$$

$$M_{4,5} = \begin{pmatrix} \mathbb{C} & \mathbb{C} & 0 & 0 & 0 & 0 & 0 \\ 0 & \mathbb{C} & \mathbb{C} & \mathbb{C} & 0 & 0 & 0 \\ 0 & 0 & \mathbb{C} & 0 & \mathbb{C} & 0 & 0 \\ 0 & 0 & 0 & \mathbb{C} & \mathbb{C} & \mathbb{C} & 0 \\ 0 & 0 & 0 & 0 & 0 & \mathbb{C} & \mathbb{C} \end{pmatrix} \quad (60)$$

We write these matrices using the basis order suggested by the graph (54). The analogous matrices for $\Lambda(A^\dagger)$ are the transpose of these.

The commutation relation

$$\Lambda(A)\Lambda(A^\dagger) \oplus \mathbf{1} \cong \Lambda(A^\dagger)\Lambda(A) \quad (61)$$

must be true in **2Vect**, since Λ preserves all the relevant structures, and since we have already demonstrated the corresponding combinatorial fact (24) about the composition of spans. This fact can be directly verified for actions on small symmetric groups using the matrices given above.

The blocks $M_{n,n+1}$ each act only on representations supported on single objects n . They represent induction functors such as

$$\text{Ind}_n^{n+1} = (+1)|_{\mathbf{n}} : \text{Rep}(S_n) \rightarrow \text{Rep}(S_{n+1}), \quad (62)$$

which induces a representation from S_n to S_{n+1} along the inclusion $S_n \subset S_{n+1}$. The adjoint blocks describe the restriction functors, showing how an S_{n+1} representation restricts to an S_n representation.

Khovanov describes a category \mathcal{S}' whose basic objects are the $\mathbb{C}[S_n] - \mathbb{C}[S_{n+1}]$ bimodules such that the tensor product over $\mathbb{C}[S_n]$ with a given representation R of S_n is the representation $M_{n,n+1}(R)$ of S_{n+1} . A general object in this category is a tensor product of a sequence of such bimodules. This is described in terms of groups, rather than the groupoid \mathbf{S} , so it is constructed as the direct sum of a collection of such categories \mathcal{S}_n , labelled by the starting value of n . These objects all represent functors which are composites of sequences of blocks like the $M_{n,n+1}$ above, or their adjoints. Taking the direct sum gives exactly the bimodules corresponding to the functors in $\text{Hom}(\text{Rep}(\mathbf{S}), \text{Rep}(\mathbf{S}))$ which are generated in this way. Thus, the image in **2Vect** of our groupoidified Heisenberg algebra corresponds exactly to this representation \mathcal{S}' of \mathbf{H}' .

3.4 Symmetrizers and antisymmetrizers

The algebra categorified by Khovanov is larger than the single-variable Heisenberg algebra described in the introduction. It is a more complicated but closely related algebra also known as the Heisenberg Vertex Algebra, isomorphic as an ordinary algebra to the infinitely-generated Heisenberg algebra. It has a countably infinite family of generators a_i (which commute amongst themselves) and their adjoints (which also commute amongst themselves). They satisfy the following relations:

$$\mathbf{a}_i \mathbf{a}_j^\dagger - \mathbf{a}_j^\dagger \mathbf{a}_i = \mathbf{a}_{j-1}^\dagger \mathbf{a}_{i-1} \quad (63)$$

This includes the original relation when $i = j = 1$, if we take $\mathbf{a}_0 = \mathbf{a}_0^\dagger = 1$. This larger algebra has a physical interpretation in terms of conformal field theory, in which operators are replaced by holomorphic operator-valued fields on a Riemann surface. This will not concern us here, though see lecture notes by Thomas [25] for a starting point.

The categorification of this larger algebra uses a larger monoidal category $\mathbf{H} = \text{Kar}(\mathbf{H}')$, which is the Karoubi envelope of \mathbf{H}' . Taking the Karoubi envelope is a universal construction which introduces new objects so that every idempotent morphism splits. In particular, \mathbf{H} contains ‘symmetrizer’ and ‘antisymmetrizer’ subobjects associated to the action of S_n on products of the form A^n and $(A^\dagger)^n$, called S_\pm^n and \bigwedge_\pm^n respectively. These summands do not exist as spans of groupoids, where the direct sum is just the disjoint union of spans, and as a result this aspect cannot be groupoidified. They do appear, however, after we apply the 2-functor Λ , since $\mathbf{2Vect}$ is complete.

In other words, this larger categorification does not appear at the level of our combinatorial model, but it can be found quite naturally by taking a completion within its image under Λ . We will consider here what this looks like in concrete terms within $\mathbf{2Vect}$.

The natural isomorphism

$$\Lambda(A) \circ \Lambda(A) \xrightarrow{\Lambda(\hat{\sigma})} \Lambda(A) \circ \Lambda(A) \quad (64)$$

that permutes the order of the operators $\Lambda(A)$ can be represented in matrix form as a collection of linear maps, each acting on a component of the matrix representation of $\Lambda(A) \circ \Lambda(A)$. That is, each of the vector spaces that appear in this matrix carries a particular action of the symmetric group S_2 . The symmetrizer operation selects the subspace which is fixed by this action, and the antisymmetrizer operation selects the complement of this subspace.

More generally, each component in A^n is a representation of S_n , which decomposes into irreducibles, and the symmetrizer takes the sub-representation consisting of exactly the copies of the trivial representation which appear in this decomposition. Similarly, the antisymmetrizer takes only the copies of the sign representation. Indeed, for any irreducible representation of S_n , there will be a correspondingly ‘symmetrized’ sub-functor of A^n - these operations are the ‘Young symmetrizers’ associated to the Young diagrams which index such representations.

In particular, the subobjects which are needed to produce all the generators of the Heisenberg algebra are those arising from the symmetrizer idempotent for $\Lambda(A^n)$ or $\Lambda((A^\dagger)^n)$ which come from the symmetric group action

$$v \mapsto \sum_{\mu \in S_n} \hat{\mu}(v) \quad (65)$$

and the antisymmetrizer

$$v \mapsto \sum_{\mu \in S_n} \text{sign}(\mu) \hat{\mu}(v). \quad (66)$$

We can illustrate this by considering the product $\Lambda(A \circ A) \cong \Lambda(A) \circ \Lambda(A)$. The nontrivial permutation σ of the 2-element set gives the 2-cell

$$A \circ A \xrightarrow{\hat{\sigma}} A \circ A, \quad (67)$$

and thus (combined with the natural isomorphism above), we obtain the natural isomorphism (64).

The composite $\Lambda(A) \circ \Lambda(A)$ is zero except for blocks of the form

$$M_{i,i+2}^2 = M_{i,i+1} \otimes M_{i+1,i+2}. \quad (68)$$

From here on, we will use the notation $M_{i,j}$ to denote the matrix of vector space which is the (i,j) th block entry of the matrix $\Lambda(A)^{j-i}$. Up to a natural isomorphism, these can be expressed in matrix form:

$$M_{0,2} = \begin{pmatrix} \mathbb{C} & \mathbb{C} \end{pmatrix} \quad (69)$$

$$M_{1,3} = \begin{pmatrix} \mathbb{C} & \mathbb{C}^2 & \mathbb{C} \end{pmatrix} \quad (70)$$

$$M_{2,4} = \begin{pmatrix} \mathbb{C} & \mathbb{C}^2 & \mathbb{C} & \mathbb{C} & 0 \\ 0 & \mathbb{C} & \mathbb{C} & \mathbb{C}^2 & \mathbb{C} \end{pmatrix} \quad (71)$$

As with the basic blocks $M_{n,n+1}$, a general block $M_{i,j}$ can be interpreted as giving a decomposition in matrix form of the $\mathbb{C}[S_i]-\mathbb{C}[S_j]$ -bimodule which effects the induction functor

$$\text{Rep}(S_i) \xrightarrow{\text{Ind}_{S_i}^{S_j}} \text{Rep}(S_j) \quad (72)$$

associated to the natural inclusion of S_i into S_j . That is, given a representation $\rho \in \text{Rep}(S_i)$, the induced representation of S_j is $\rho \otimes_{\mathbb{C}[S_i]} M_{i,j}$.

As an example, the block $M_{2,4}$ has component \mathbb{C}^2 in the position indexed by $(\square\square, \begin{smallmatrix} \square & \square \\ \square & \square \end{smallmatrix})$. As we have remarked, this reflects the fact that there are two paths in the directed graph (54) from $\square\square$ to $\begin{smallmatrix} \square & \square \\ \square & \square \end{smallmatrix}$, passing through $\square\square\square$ or $\begin{smallmatrix} \square & \square \\ \square & \square \end{smallmatrix}$.

A Young tableaux is a way of filling the boxes of the diagram with the numbers $1, \dots, n$ such that they denote a valid order in which to add the boxes while following a path through (54). It is a standard fact that the tableaux which can fill a given Young diagram label basis elements of the irreducible representation labelled by that diagram. Thus, we see that the component \mathbb{C}^2 acts to exchange the two basis elements. This is a permutation representation on the basis of \mathbb{C}^2 , which decomposes as:

$$\square\square \oplus \begin{smallmatrix} \square \\ \square \end{smallmatrix} \quad (73)$$

The symmetrizer and antisymmetrizer each select just one of these summands. The two other components are both trivial S_2 representations. So the symmetrized functor has a matrix form with a $(2,4)$ block:

$$\begin{pmatrix} \mathbb{C} & \mathbb{C} & \mathbb{C} & \mathbb{C} & 0 \\ 0 & \mathbb{C} & \mathbb{C} & \mathbb{C} & \mathbb{C} \end{pmatrix} \quad (74)$$

The antisymmetrized functor has the following block:

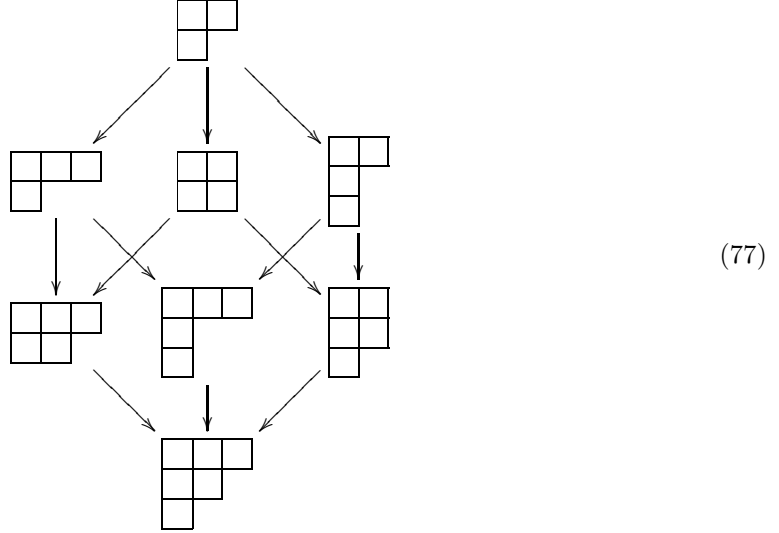
$$\begin{pmatrix} 0 & \mathbb{C} & 0 & 0 & 0 \\ 0 & 0 & 0 & \mathbb{C} & 0 \end{pmatrix} \quad (75)$$

The situation becomes more complex for A^n with larger values of n . To illustrate this, the next example shows the first case where there are three new boxes to add, and the

order is unconstrained. Consider the product $\Lambda(A^3) \cong \Lambda(A)^3$, and in particular look at the block

$$M_{3,6} = \begin{pmatrix} \mathbb{C} & \mathbb{C}^3 & \mathbb{C}^3 & \mathbb{C}^3 & \mathbb{C} & \mathbb{C}^2 & 0 & \mathbb{C} & 0 & 0 & 0 \\ 0 & \mathbb{C} & \mathbb{C}^3 & \mathbb{C}^3 & \mathbb{C}^2 & \mathbb{C}^6 & \mathbb{C}^2 & \mathbb{C}^3 & \mathbb{C}^3 & \mathbb{C} & 0 \\ 0 & 0 & 0 & \mathbb{C} & 0 & \mathbb{C}^2 & \mathbb{C} & \mathbb{C}^3 & \mathbb{C}^3 & \mathbb{C}^3 & \mathbb{C} \end{pmatrix} \quad (76)$$

This can be easily checked by calculation, but in particular, the component indexed by $\begin{smallmatrix} \square & \square \\ \square & \end{smallmatrix}$ and $\begin{smallmatrix} \square & \square & \square \\ \square & \square & \end{smallmatrix}$ is \mathbb{C}^6 , which reflects the fact that there are six ways to add the three new blocks, corresponding to paths through this directed graph:



By an analogous argument to the one in the previous example, this becomes a representation of S_3 by permutations of the three new boxes, and hence the axes of the cube above. The basis for the component \mathbb{C}^6 consists of the labellings of the three boxes by numbers $\{4, 5, 6\}$. These can be appended to any Young tableau in $\begin{smallmatrix} \square & \square \\ \square & \end{smallmatrix}$, and thereby generate a basis for the tensor product with the bimodule which $M_{3,6}^3$ depicts. Thus, S_3 acts by permutations on the set of orders of $\{4, 5, 6\}$. This is equivalent, as an S_3 representation, to the action on the regular representation $\mathbb{C}[S_3]$ itself. This decomposes as

$$\begin{smallmatrix} \square & \square & \square \end{smallmatrix} \oplus 2 \begin{smallmatrix} \square & \square \\ \square & \end{smallmatrix} \oplus \begin{smallmatrix} \square \\ \square \\ \square \end{smallmatrix} \quad (78)$$

So again, we have a one-dimensional subspace in this position for either the symmetrized or antisymmetrized product. This block is the first case where the other Young symmetrizers, in this case the one associated to $\begin{smallmatrix} \square & \square \\ \square & \end{smallmatrix}$, will give a nontrivial component.

We have seen here that the functors $\Lambda(A)$ and $\Lambda(A^\dagger)$ have, as sub-functors, precisely the symmetric and antisymmetric powers S_\pm^n and \bigwedge_\pm^n which will give the actual generators of the multivariable Heisenberg algebra when passing to the Grothendieck ring. So, after taking the image under Λ of our groupoidification, passing to the completion of the image recovers Khovanov's full categorification. This is closely related to a construction used by Khovanov in the proof that there is a map $H_{\mathbb{Z}}$ from the integral form of the Heisenberg algebra into the Grothendieck group of \mathbf{H} .

4 Combinatorial models of categorified $U(\mathfrak{sl}_n)$

4.1 Introduction

We now investigate how the techniques described so far can be applied to find combinatorial models of categorifications of $U(\mathfrak{sl}_n)$, the universal enveloping algebras of the Lie algebras \mathfrak{sl}_n .

These categorifications are in the style of the Khovanov-Lauda programme [14, 17, 16]. The main objects of interest in that programme are categorifications of the q -deformed enveloping algebras, the quantum groups $U_q(\mathfrak{sl}_n)$. However, the combinatorial interpretation we describe here only applies to the case $q = 1$, in which case the Grothendieck ring is a module over the integers \mathbb{Z} . In the q -deformed situation, one might expect to obtain a module over the ring $\mathbb{Z}[q, q^{-1}]$ of formal power series in q .

We expect that a similar combinatorial model should exist of the categorification of the full q -deformed algebra, following the pattern of the groupoidification of Hecke algebras [2]. In this approach, the groupoid of finite sets is replaced with the groupoid of finite-dimensional vector spaces over a finite fields \mathbb{F}_q . The special linear groups for such vector spaces take the role of symmetric groups S_n , and their size is related to q -factorials.

4.2 The algebras $U(\mathfrak{sl}_n)$

We write \mathfrak{sl}_n for the Lie algebra associated to the special linear group $SL(n)$, which is given as the group of $n \times n$ matrices with unit determinant. Its Lie algebra would then consist of the traceless $n \times n$ matrices. It is presented as an abstract algebra in terms of generators h_i , e_i and f_i for $i \in \{1, \dots, n-1\}$, satisfying the following relations:

$$[e_i, f_j] = \delta_{ij} h_i \quad (79)$$

$$[e_i, h_j] = 2\delta_{ij} e_i \quad (80)$$

$$[f_i, h_j] = 2\delta_{ij} f_i \quad (81)$$

This is a special case of the standard presentation, in the Chevalley basis, of the Lie algebra generated by Cartan data associated to a particular Dynkin diagram [10, 8]. In the representation of \mathfrak{sl}_2 as the traceless matrices acting on \mathbb{C}^2 , these elements have the following concrete realization:

$$e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \quad f = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \quad h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad (82)$$

Just as the Heisenberg algebra has a natural representation on the polynomial ring $\mathbb{C}[[z]]$, the universal enveloping algebras $U(\mathfrak{sl}_n)$ have natural representations on polynomial rings $\mathbb{C}[[z_1, \dots, z_n]]$, defined in the following way:

$$\mathbb{C}[[z_1, \dots, z_n]] = \bigoplus_j (\mathbb{C}^n)^{\otimes_s j} \quad (83)$$

In this polynomial representation, the generators e_i and f_i of $U(\mathfrak{sl}_n)$ become

$$e_i = z_{i+1} \partial_i \quad (84)$$

$$f_i = z_i \partial_{i+1}, \quad (85)$$

and their commutator is

$$h_i = z_{i+1} \partial_{i+1} - z_i \partial_i. \quad (86)$$

This extension gives back the defining representation of \mathfrak{sl}_n when \mathbb{C}^n is identified with the space of degree-1 polynomials.

This representation of $U(\mathfrak{sl}_n)$ suggests defining

$$n_i := z_i \partial_i, \quad (87)$$

so that we have

$$h_i = n_{i+1} - n_i. \quad (88)$$

Rewriting our Lie algebra relations (84–86) in terms of these new variables, and expanding out commutators explicitly and reorganizing to eliminate negatives, we obtain the following:

$$e_i f_j + \delta_{ij} n_i = f_j e_i + \delta_{ij} n_{i+1} \quad (89)$$

$$e_i n_{j+1} + n_j e_i = e_i n_j + n_{j+1} e_i + 2\delta_{ij} e_i \quad (90)$$

$$f_i n_{j+1} + n_j f_i = f_i n_j + n_{j+1} f_i + 2\delta_{ij} f_i \quad (91)$$

In this form, the relations are now suitable for groupoidification. We note that these relations are a special case of the general pattern for the construction of a Kac-Moody algebra from a Cartan matrix. In the case of these Lie algebras, the Cartan matrix is determined by the Dynkin diagram associated to the algebra. So the pattern for groupoidifying these algebras should apply in these more general cases as well.

4.3 Groupoidification

The groupoidification of \mathfrak{sl}_n is based on the groupoid of n -coloured finite sets and colour-preserving bijections, written \mathbf{S}^n . This is equivalent to the category where objects are n -tuples of finite sets and morphisms are n -tuples of bijections, which is the cartesian product of n copies of \mathbf{S} . The sets of colour i form the i th component of the cartesian product. A third way to describe \mathbf{S}^n up to equivalence is as the free symmetric monoidal groupoid on the discrete groupoid with n objects.

We write $\mathbf{S}^n \xrightarrow{+1_i} \mathbf{S}^n$ for the functor which acts as $+1$ on the sets of colour i , and the identity otherwise. For any $i \in \{1, \dots, n\}$, we use this to define the spans A_i and A_i^\dagger as follows:

$$\begin{array}{ccc} & \mathbf{S}^n & \\ \text{id} \swarrow & & \searrow +1_i \\ \mathbf{S}^n & & \mathbf{S}^n \end{array} \qquad \begin{array}{ccc} & \mathbf{S}^n & \\ +1_i \swarrow & & \searrow \text{id} \\ \mathbf{S}^n & & \mathbf{S}^n \end{array} \quad (92)$$

These are ‘multicoloured’ variants of the ordinary annihilation and creation spans defined in equation (10). They satisfy an adjusted version of the categorified commutation relation:

$$A_i \circ A_j^\dagger \simeq A_j^\dagger \circ A_i \oplus \delta_{ij} \text{id}_{\mathbf{S}^n} \quad (93)$$

Here δ_{ij} represents the identity span if $i = j$, and the zero span otherwise.

To obtain our model of categorified $U(\mathfrak{sl}_n)$, we arbitrarily assign a total ordering to our set n of colours. We then make the following definitions for all $i \in \{1, \dots, n-1\}$:

$$E_i := A_{i+1}^\dagger \circ A_i \quad (94)$$

$$F_i := A_i^\dagger \circ A_{i+1} \quad (95)$$

These are mutually-converse spans for each i . We also define the number operator N_i for each colour i as follows:

$$N_i := A_i^\dagger \circ A_i \quad (96)$$

This span is its own converse. The operators E_i , F_i and N_i satisfy a categorified form of the reorganized relations (89–91) for $U(\mathfrak{sl}_n)$:

$$E_i F_j \oplus \delta_{ij} N_i \simeq F_j E_i \oplus \delta_{ij} N_{i+1} \quad (97)$$

$$E_i N_{j+1} \oplus N_j E_i \simeq E_i N_j \oplus N_{j+1} E_i \oplus 2\delta_{ij} E_i \quad (98)$$

$$F_i N_{j+1} \oplus N_j F_i \simeq F_i N_j \oplus N_{j+1} F_i \oplus 2\delta_{ij} F_i \quad (99)$$

The reason for this is combinatorial, and can be understood by explicitly tracking histories. For instance, in the first relation, consider the span $E_i F_j$. This is equal to $A_j^\dagger \circ A_i \circ A_i^\dagger \circ A_j$, a sequence of adding and removing elements of colours i and j . In the case where $i \neq j$ the operators A_i and A_j commute, so this is isomorphic to $A_i \circ A_i^\dagger \circ A_j^\dagger \circ A_j$. Since the spans A_i and A_j each satisfy the usual categorified commutation relations for the Heisenberg algebra, this is isomorphic to $(N_i \oplus \text{id}) \circ N_j \simeq N_i N_j \oplus N_j$. A similar computation holds on the other side, and so the isomorphism (97) can be established. Applied to a coloured set with n_i elements of colour i , and n_j elements of colour j , both sides compute $n_i n_j + n_i + n_j$. If we add the identity on each side, this relation for E_i and F_j witnesses the isomorphism of two different ways to count $(n_i + 1)(n_j + 1)$, the number of ways to distinguish either zero or one elements of each colour in any given set.

Applying the degroupoidification functor (47) to E_i , F_i and N_i recovers the standard action (79–81) of $U(\mathfrak{sl}_n)$ on spaces of polynomials. The coefficients obtained by the multiplication and differentiation are exactly the numbers we have just calculated in the example. Just as with the Heisenberg algebra, the algebraic facts about the relations they satisfy now appear as consequences of combinatorial facts about coloured sets. In this way, we have a groupoidification of the concrete representation on $\mathbb{C}[[z_1, \dots, z_n]]$ of the algebra $U(\mathfrak{sl}_n)$. In the same way, the abstract categorifications of Khovanov-Lauda type are realized concretely by endomorphisms of an object in **Span(Gpd)**.

We can make a further comment about this groupoidification from a physical point of view. As described in Section 2.2, the groupoidification of ‘states’ in Fock space for the quantum harmonic oscillator can be described as stuff types, a generalization of combinatorial species. That is, such a state is a groupoid \mathbf{X} equipped with a functor $\mathbf{X} \xrightarrow{\Phi} \mathbf{S}$. We think of \mathbf{X} as an ensemble of structures whose underlying finite sets are specified by Φ , and which have a notion of symmetry that is preserved by Φ . Combinatorial species are exactly the case where Φ is faithful, meaning that morphisms of \mathbf{X} are determined by the underlying bijection of finite sets. Considering a stuff type as a span $\mathbf{X} \xleftarrow{\Phi} \mathbf{S} \rightarrow \mathbf{1}$ and degroupoidifying it, we obtain a linear map $\mathbb{C} \rightarrow \mathbb{C}[[z]]$, which specifies a power series in z by considering the image of $1 \in \mathbb{C}$. This is the generating function in the ordinary sense for the combinatorial structure [27].

In the same way, a state in the Fock space for our groupoidified $U(\mathfrak{sl}_n)$ can be described as a groupoid \mathbf{X} equipped with $\mathbf{X} \xrightarrow{\Psi} \mathbf{S}^n$, thought of as a span out of $\mathbf{1}$. This is a generalization of a *multisort species* [3] which describe structures on coloured sets, in the same way that stuff types generalize species. Thus, we can describe such a state as a *multisort stuff type*. Again, degroupoidification of such a span determines an element of $\mathbb{C}[[z_1, \dots, z_n]]$, so a multisort stuff type determines a vector in the multiple-variable Fock space. The groupoidified $U(\mathfrak{sl}_n)$ acts on multisort stuff types just as the groupoidified Heisenberg algebra acts on stuff types.

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